Function

The great mathematician Leibnitz have coined the word function in 1694. The functional notation $f(x)$ was invented by Euler in 1734.

Definition

A relation f from X to Y is called a 'function' if it satisfies the following two conditions:

(i) $D_f = \text{dom} f = X$ (ii) $(x,y) \in f$ and $(x,z) \in f \Rightarrow y = z$

Thus a function from X to Y is a relation whose domain is the whole of X and is not one-many.

Some of the English synonyms for the word function are:

mapping, map, transformation, transform, operator, correspondence.

If $f \subset X \times Y$ is a function, we often write this as $f: X \to Y$ and say that f is a function from X to Y, or on X to Y or f maps X into Y.

If $(x,y) \in f$, then we write $y = f(x)$.

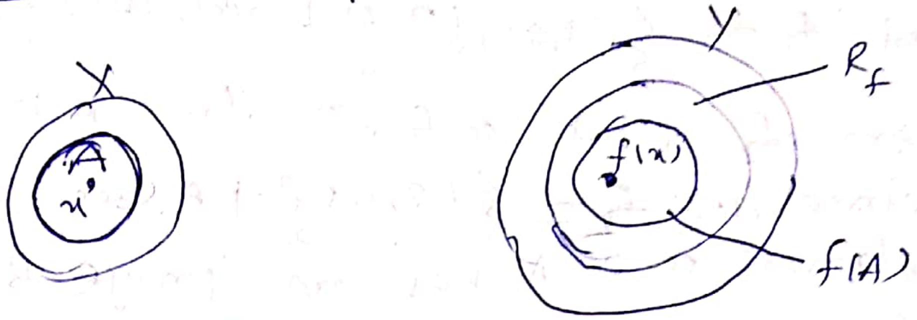
The set Y is called codomain of f. It is evident that $R_f = \text{rng}(f) \subseteq Y$

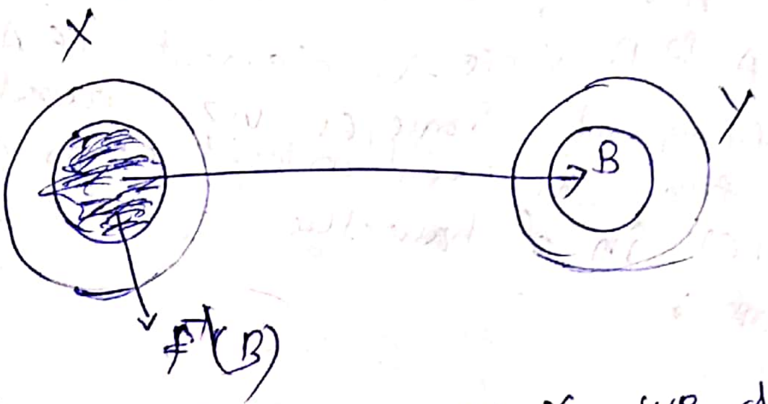
~~and this~~ For any subset A of X, the image of A under f is the set ~~f(A) =~~ $f(A) = \{f(x) : x \in A\}$ and for any $B \subseteq Y$, the pre-image of B is the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$

The range of f is defined by

$R_f = rng(f) = f(X) =$ the set of all images of elements of X under f.

<u>Diagrammatic Representation of a function:</u>





By varying x over X, we determine f(x) for each $x \in X$ and so the func^n. $f: X \to Y$ can also be written as

$$f = \{(x, f(x)) : x \in X\}$$

and x is called the variable.

<u>Exa:</u> Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c\}$

consider $f_1 = \{(1,a), (2,b), (3,b), (5,b)\}$

Obviously $f_1 \subset A \times B$

Here dom $f_1 = \{1, 2, 3, 4, 5\} = A$

Codomain of $f_1 = \{a, b, c\} = B$

rng $f_1 = \{a, b\} \subset B$

Moreover every element of A has a unique image in B, Hence f_1 is a func. from A to B.

Now for the above sets A & B consider

$f_2 = \{(1,b), (2,a), (3,c), (5,a)\}$

and $f_3 = \{(1,a), (2,b), (3,a), (4,b), (5,c), (1,c)\}$

Here f_2 is not a func. from A to B since dom $f_2 = \{1,2,3,5\} \neq A$ as the element 4 of A has no image B.

Also f_3 is not a function from A to B since the element $1 \in A$ has two different images viz. a and c i.e., two different ordered pairs (1,a) and (1,c) in f have the same first component.

Some real functions and their graphs.

Example 14

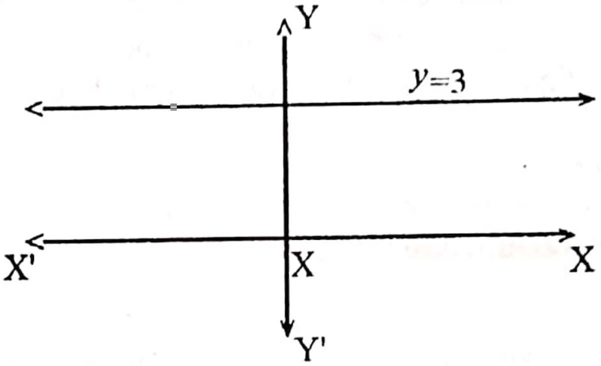
Constant function :

A function $f : A \to R$ is said to be a constant function if these is a real number k such that $f(x) = k$, for all $x \in A$.

Here dom $f = A \subseteq R$, $rng f = \{k\}$ which is a singleton.

The function $f = \{(0,1), (\sqrt{2}, 1), (-\frac{1}{3}, 1)\}$ is a constant function with domain $\{0, \sqrt{2}, -\frac{1}{3}\}$ and range $\{1\}$.

The graph of the constant function $f(x) = 3, x \in R$ is the line parallel to x - axis as shown:



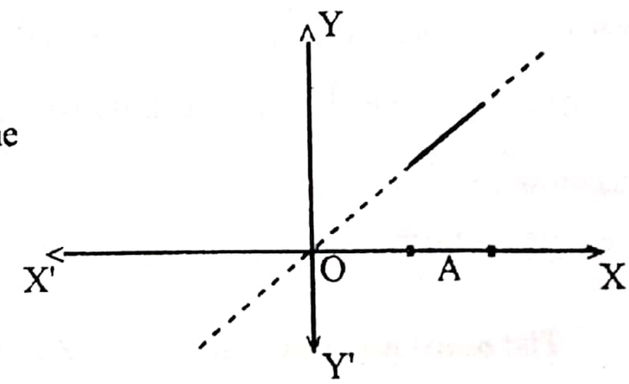
Example 15

Identity Function :

For any nonempty set $A \subseteq R$ the function $f : A \to A$ defined by $f(x) = x$ for all $x \in A$ is called the identity function on A. It is denoted by id_A.

For the identity function f, dom $f = rng f$.

The graph of id_A on $A \subseteq R$ is part of the straight line through the origin as shown.



(Here A is supposed to be an interval as shown in the figure.)

Exponential function :

An **exponential function** is defined by $f(x) = a^x$ $(a > 0,\ a \neq 1),\ x \in R$. The fact that a^x exists for every $x \in R$ whenever $a > 0$ follows from the theory of real numbers. The following properties can also be proved.

(a) $a^{x+y} = a^x a^y,\ a > 0$

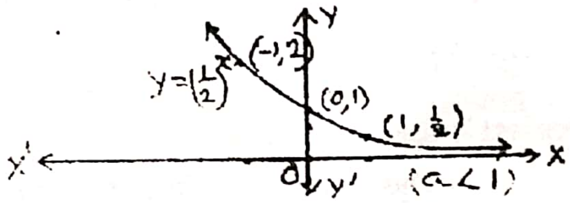
$(a^x)^y = a^{xy},\ x, y \in R$

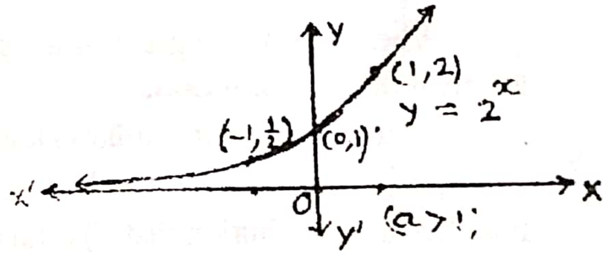
(b) $a^x = 1$ iff $x = 0$

(c) If $a > 1$, $a^x > a^y$ iff $x > y$.

(d) If $a < 1$, then $a^x > a^y$ iff $x < y$

It follows that the exponential function as defined above is one - to - one and monotonic (increasing or decreasing according as $a > 1$ or $a < 1$).





(e) a^x is closer to the X - axis as x recedes away from zero along negative values.

The graph of $f(x) = y = 2^x$ is shown in Graphs of $y = 3^x$, $y = 4^x$ etc. can be similarly plotted, but the growth is so rapid that for even for a value like $a = 4$, the graph connot be accommodated on the space provided here for $x \geq 2$.

A comparision with the graphs of $y = x^2$ or $y = x^3$ will show that 2^x grows much more rapidly than x^2 or x^3 or indeed than x^n for any n.

It is clear that dom f = R and rng f = R^+.

Example 21

Logarithmic function

The function f defined by $f(x) = \log_a x$, $(a > 0,\ a \neq 1)$ where $y = \log_a x \Leftrightarrow a^y = x$ is called the logarithmic function to the base a.

Here dom $\log_a = R^+$, rng $\log_a = R$.

Properties of logarithmic function which can be easily derived from definition are given below:

(i) $\log_a(xy) = \log_a x + \log_a y$

(ii) $\log_a(x/y) = \log_a x - \log_a y$

(iii) $\log_a x = 0 \Leftrightarrow x = 1$

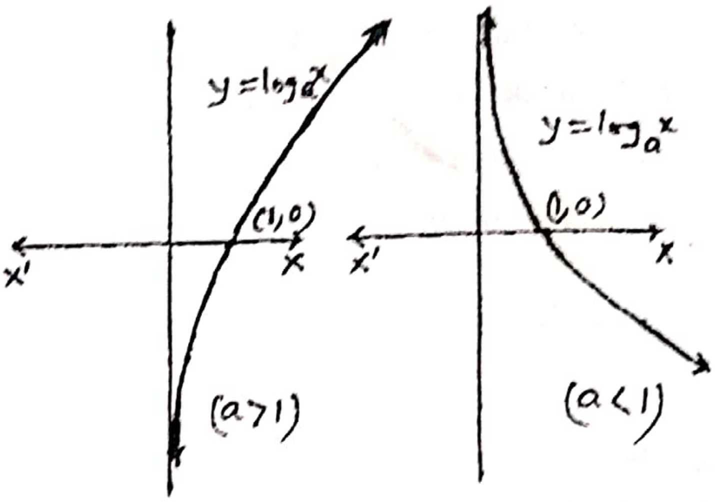
(iv) $\log_x x = 1$

(v) $\log_a x = 1/\log_x a,\ x \neq 1$

(vi) $\log_a x = \log_b x \cdot \log_a b$.

(vii) If $a > 1$, $\log_a x > \log_a y$ iff $x > y$

and if $a < 1$, $\log_a x < \log_a y$ iff $x > y$



(viii) $\dfrac{\log_a x}{\log_a y} = \log_y x$, $(y \neq 1)$

The accompanying figures show the graphs of $\log_a x$ *for* $a > 1$ and $a < 1$ respectively. Both the graphs meet the x- axis at (1, 0) and never meet the y- axis.

Example 22

Greatest Integer function

The function f defined by $f(x) = [\,x\,]$

Where $[\,x\,]$ is the greatest integer not greater than x (less than or equal to x) is called the **greatest integer function.**

From the difinition if follows that if n is an integer,

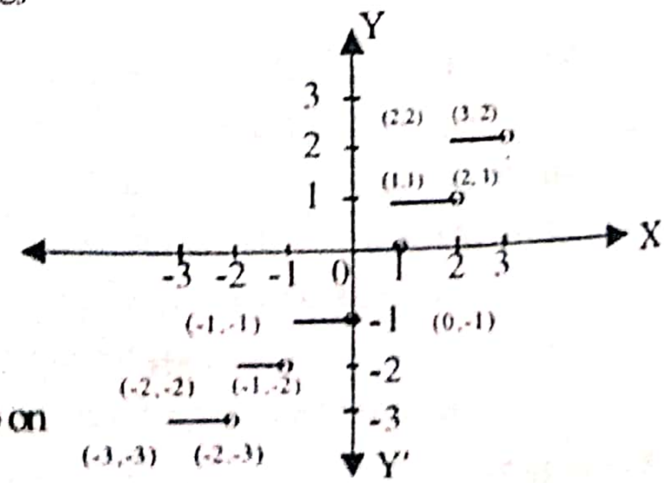
$[\,x\,] = n$ for $n \le x < n + 1$

It is clear from definition that (i) dom$f = R$, (ii) rng$f = Z$

$$\text{So } f(x)\,[\,x\,] = \begin{cases} 0 \text{ if } 0 \le x < 1 \\ 1 \text{ if } 1 \le x < 2 \\ 2 \text{ if } 2 \le x < 3 \text{ and so on.} \end{cases}$$

and

$$\begin{cases} -1 \text{ if } -1 \le x < 0 \\ -2 \text{ if } -2 \le x < -1 \\ -3 \text{ if } -3 \le x < -2 \text{ and so on} \end{cases}$$



The graph of $y = f(x) = [\,x\,]$ is plotted above

The graph consists of infinitely many closed open parallel line segments

This graph looks like steps.

(ii) Transcendental functions

All functions which are not algebraic are called transcendental functions. Examples of such functions are given below.

(a) Trigonometric function :

sine : $R \to [-1, 1]$

cosine : $R \to [-1, 1]$

tangent : $R' \to R$ Where $R' = R - \{(2n+1)\frac{\pi}{2} : n \in Z\}$

cotangent : $R'' \to R$ where $R'' = R - \{n\pi : n \in Z\}$

secant : $R' \to R$

cosecant : $R'' \to R$

The above trigonometric functions are abbreviated as sin, cos, tan, cot, sec and cosec respectively. Note that the domains of tan, cot, sec and cosec could not be all of R but a truncated R

(b) Inverse Trigonometric functions

These functions shall come up in Vol-II.

(c) Exponential functions

Functions of the type a^x, $a > 0$ and $a \neq 1$, e^x etc are called exponential functions. More generally functions of the type $x^{\sin x}$, $(\cos x)^{\log x}$ are also transcendental.

1. Give an example of a relation which is not a fuction.

2. If X and Y are sets containing m and n elements respectively then what is the total number of fuctions from X to Y ? 8.

3. Find the domain of the following functions :

(i) $\sqrt{9-x^2}$ (ii) $\frac{x}{1+x^2}$ (iii) 1 - | x | (iv) $\frac{1}{x^2-1}$

(v) $\frac{\sin x}{1+\tan x}$ (vi) $\frac{x}{|x|}$ (vii) $\frac{1}{x+|x|}$ (viii) $\sqrt{\log\frac{12}{(x^2-x)}}$

9

Limits and Continuity

Limit of a Function

Definition

A number l is called the limit of a function f as x tends to a or simply $\lim_{x \to a} f(x) = l$ if for any $\epsilon > 0$ there exists $\delta > 0$ depending on ϵ such that

$0 < |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon$,

i.e. $a-\delta < x < a+\delta$ and $x \neq a \Rightarrow l-\epsilon < f(x) < l+\epsilon$.

Exa:1

$\lim_{x \to a} x = a$

Sol^n:

Let ϵ be any positive number.

We take $f(x) = x$

Then $|f(x) - a| < \epsilon$ if $|x-a| < \epsilon$.

Taking $\delta = \epsilon$ we see that there exists $\delta > 0$ depending on ϵ such that

$|x-a| < \delta \Rightarrow |f(x) - a| < \epsilon$

So $0 < |x-a| < \delta \Rightarrow |f(x) - a| < \epsilon$

Hence $\lim_{x \to a} f(x) = a$

i.e. $\lim_{x \to a} x = a$

Exa Show that $\lim_{x \to 1} (2x+1) = 3$

Sol^n:

Let ϵ be any positive number.

Then $|(2x+1) - 3| < \epsilon$ if $2|x-1| < \epsilon$,

i.e. if $|x-1| < \frac{\epsilon}{2}$.

We set $\delta = \frac{\epsilon}{2}$. Thus there exist

$\delta > 0$ depending on ϵ such that $|x-1| < \delta$
$\Rightarrow |(2x+1) - 3| < \epsilon$.

So $\lim_{x \to 1} (2x+1) = 3$.

Questions

(1) Use intuition and then $\epsilon-\delta$ technique to obtain the following limits.

(1) $\lim_{x \to 1} (4x-1)$

(2) $\lim_{x \to 3} \frac{x^3-9}{x-3}$

(3) $\lim_{x \to 1} (\sqrt{x} + 3)$

(4) $\lim_{x \to 1} \frac{3x+2}{2x+3}$

Laws of limits :

The following theorem is useful for evaluation of limits. Different parts of this theorem can be proved by using the $\epsilon - \delta$ definition of the limit.

Theorem 1 :

If the functions f and g are such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then

(i) $\lim_{x \to a} \{f(x) + g(x)\} = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = l + m,$

i.e. the limit of a sum is equal to the sum of the limits.

(ii) $\lim_{x \to a} \{f(x) - g(x)\} = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = l - m,$

i.e. the limit of a difference is equal to the difference of the limits.

(iii) $\lim_{x \to a} \{k f(x)\} = k \lim_{x \to a} f(x) = kl$

if k is a constant.

(iv) $\lim_{x \to a} \{f(x) g(x)\} = \lim_{x \to a} f(x) . \lim_{x \to a} g(x) = lm,$

i.e. the limit of a product is equal to the product of the limits.

(v) $\lim_{x \to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$ provided $m \neq 0,$

i.e. the limit of a quotient is equal to the quotient of the limits when the limit of the denominator is nonzero.

Example 8 :

(i) $\lim_{x\to 1}\left(\sqrt{x} + x + \frac{1}{\sqrt{x}}\right)$

$= \lim_{x\to 1}\left(\sqrt{x}\right) + \lim_{x\to 1}(x) + \lim_{x\to 1}\frac{1}{\sqrt{x}}$ by theorem 1 (i)

$= 1 + 1 + 1 = 3.$

(ii) $\lim_{x\to 1}(17\sqrt{x}) = 17\lim_{x\to 1}\sqrt{x} = 17.$ by theorem 1 (iii)

(iii) $\lim_{x\to 0}(2x+1)(\sqrt{x}+5)$

$= \lim_{x\to 0}(2x+1)\lim_{x\to 0}\left(\sqrt{x}+5\right)$ by theorem 1 (iv)

$= 1 . 5 = 5.$

(iv) $\lim_{x\to 1}\left(\frac{x+\sqrt{x}}{2x+1}\right) = \left(\frac{\lim_{x\to 1}\left(x+\sqrt{x}\right)}{\lim_{x\to 1}(2x+1)}\right) = \frac{1+1}{2+1} = \frac{2}{3}.$ by theorem 1 (v)

Scanned with CamScanner

Definition :

A number l_1 is called the left-hand limit of $f(x)$ at $x = a$ or simply $\lim_{x \to a-} f(x) = l_1$ if for any $\in > 0$ there exists $\delta > 0$ depending on $\in$ such that

$$a - \delta < x < a \Rightarrow |f(x) - l_1| < \in.$$

Definition :

A number l_2 is called the right-hand limit of $f(x)$ at $x = a$ or simply $\lim_{x \to a+} f(x) = l_2$ if for any $\in > 0$ there exists $\delta > 0$ depending on $\in$ such that

$$a < x < a + \delta \Rightarrow |f(x) - l_2| < \in.$$

Using the last two definitions and the definition of the limit of $f(x)$ at $x = a$ it is easy to see that

(1) if $l_1 = l_2 = l$,

i.e. $\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = l$, then

$\lim_{x \to a} f(x)$ exists and is equal to l,

i.e. $\lim_{x \to a} f(x) = l$.

(2) If $l_1 \neq l_2$, then $\lim_{x \to a} f(x)$ does not exist.

(3) For a function f to have a limit at a point $a \in R$ it is necessary and sufficient that both $\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ exist, and these limits coincide.

Definition :

$\lim_{x \to a} f(x) = \infty$ if given $G > 0$, there exists $\delta > 0$ depending on G such that $0 < | x - a | < \delta$ $\Rightarrow f(x) > G$.

Definition :

$\lim_{x \to a} f(x) = -\infty$ if given $G > 0$, there exists $\delta > 0$ depending on G such that $0 < |x - a| < \delta$ $\Rightarrow f(x) < -G$.

Note that in the preceding definitions we can choose G as large as we please.

Example 11 :

Show that $\lim_{x \to 0+} \frac{1}{x(x+1)} = \infty$ and $\lim_{x \to 0-} \frac{1}{x(x+1)} = -\infty$.

Solution :

Let G be any positive number, however large.

Then given $G > 0$, there exists $\delta > 0$ depending on G such that

if $0 < x < x(1 + x) < \frac{1}{G}$, then $\frac{1}{x(1+x)} > G$.

We take $\delta = \frac{1}{G}$.

Thus given $G > 0$, there exists $\delta > 0$ depending on G such that

$0 < x < \delta \Rightarrow \frac{1}{x(1+x)} > G$.

So $\lim_{x \to 0+} \frac{1}{x(1+x)} = \infty$.

Again for large G, $-\frac{1}{G} < x < 0$ implies $0 < x + 1 < 1$. So

$-\frac{1}{G} < x < (x+1)x < 0$ so that $\frac{1}{x(x+1)} < -G$.

Hence $\lim_{x \to 0-} \frac{1}{x(x+1)} = -\infty$.

Example 12 :

Show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution :

Let G be any positive number, however large.

Then $\frac{1}{x^2} > G$ if $x^2 < \frac{1}{G}, x \neq 0$,

i.e. if $-\frac{1}{\sqrt{G}} < x < \frac{1}{\sqrt{G}}, x \neq 0$,

i.e. if $0 < |x| < \frac{1}{\sqrt{G}}$.

We take $\delta = \frac{1}{\sqrt{G}}$.

Thus given $G > 0$, there exists $\delta > 0$ depending on G such that

$$0 < |x| < \delta \Rightarrow \frac{1}{x^2} > G.$$

So $\lim_{x \to 0} \frac{1}{x^2} = \infty.$

Definition :

$\lim_{x \to \infty} f(x) = l$ if given $\in > 0$, there

exists $G > 0$ depending on $\in$ such that

$$x > G \Rightarrow |f(x) - l| < \in.$$

Definition :

$\lim_{x \to -\infty} f(x) = l$ if given $\in > 0$, there

exists $G > 0$ depending on $\in$ such that

$$x < -G \Rightarrow |f(x) - l| < \in.$$

Definition :

$\lim_{x \to \infty} f(x) = \infty$ if given $G > 0$, there

exists $k > 0$ such that

$$x > k \Rightarrow f(x) > G.$$

The concepts $\lim_{x \to \infty} f(x) = -\infty$ and $\lim_{x \to -\infty} f(x) = \infty$ can be defined similarly.

Example 14 :

Evaluate $\lim_{x \to \infty} \frac{3x^2 + 4x - 1}{2x^2 - 3x + 5}$.

Solution :

$$\lim_{x \to \infty} \frac{3x^2 + 4x - 1}{2x^2 - 3x + 5}$$

$$= \lim_{x \to \infty} \frac{3 + \frac{4}{x} - \frac{1}{x^2}}{2 - \frac{3}{x} + \frac{5}{x^2}}$$

$$= \frac{\lim_{x \to \infty} \left(3 + \frac{4}{x} - \frac{1}{x^2}\right)}{\lim_{x \to \infty} \left(2 - \frac{3}{x} + \frac{5}{x^2}\right)}$$

$$= \frac{\lim_{x \to \infty} 3 + \lim_{x \to \infty} \frac{4}{x} - \lim_{x \to \infty} \frac{1}{x^2}}{\lim_{x \to \infty} 2 - \lim_{x \to \infty} \frac{3}{x} + \lim_{x \to \infty} \frac{5}{x^2}}$$

$$= \frac{3 + 0 - 0}{2 - 0 + 0} = \frac{3}{2}.$$

We state below the following theorems without proof. These theorems are useful for solving some problems on limits.

Scanned with CamScanner

11.5 Convergence of a sequence

Definition :

A number l is called the limit of a sequence $\{x_n\}$ if given $\epsilon > 0$, there exists $m \in N$ such that

$$|x_n - l| < \epsilon \text{ for } n > m.$$

If l is the limit of a sequence $\{x_n\}$, then we write $\lim_{n \to \infty} x_n = l$.

Definition :

A sequence $\{x_n\}$ is said to be convergent if $\lim_{n \to \infty} x_n$ exists; otherwise it is called divergent.

If $\lim_{n \to \infty} x_n = l$, then we say that the sequence $\{x_n\}$ converges to l.

Example 16 :

Prove that the sequence $\left\{\frac{1}{n}\right\}$ is convergent.

Solution :

Let $\epsilon > 0$.

Then $\frac{1}{n} < \epsilon$ if $n > \frac{1}{\epsilon}$

We can choose $m \in N$ such that $n > m > \frac{1}{\epsilon}$.

Thus there exists $m \in N$ such that

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon \text{ for } n > m.$$

So the sequence $\left\{\frac{1}{n}\right\}$ converges to 0;

i.e. $\lim_{n \to \infty} \frac{1}{n} = 0.$

Theorem 4 : (Without proof)

A sequence $\{x_n\}$ converges to l if and only if every sub-sequence of $\{x_n\}$ converges to l.

Example 18 :

The sequence $\{(-1)^n\}$ is divergent.

Solution :

Write $x_n = (-1)^n$.

Then $x_1 = -1,\ x_2 = 1,\ x_3 = -1,\ x_4 = 1, \ldots\ldots$

i.e. $x_{2k} = (-1)^{2k} = 1, x_{2x+1} = (-1)^{2k+1} = -1.$

Then the value of x_n is 1 if n is even, -1 if n is odd. In other words this sequence **oscillates**. We may see that

$$\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} 1 = 1$$

$$\lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} (-1) = -1.$$

So by Theorem 4, the sequence is divergent.

Theorem 5 : (Without Proof)

$$\lim_{x\to a} f(x) = l \Leftrightarrow \lim_{n\to\infty} f(x_n) = l$$

for every real sequence $\{x_n\}$ with $x_n \neq a$

for any $n \in \mathrm{N}$ and $\lim_{n\to\infty} x_n = a.$

Example 19 :

Prove that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Solution :

Let $f(x) = \sin\frac{1}{x},\ x \neq 0.$

Let $\{x_n\}$ be a sequence defined by

$$x_n = \frac{2}{(2n+1)\pi}\ \forall n \in \mathrm{N}.$$

clearly $x_n \neq 0$ for any $n \in \mathrm{N}$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2}{(2n+1)\pi} = 0.$$

We have $f(x_n) = \sin \frac{1}{x_n}$

$= \sin (2n+1) \frac{\pi}{2}$

$= (-1)^n \; \forall \, n \in N.$

$\{f(x_n)\}$ is a sequence in R.

Consider the subsequences $\{f(x_{2n})\}$ and $\{f(x_{2n-1})\}$ of the sequence $\{f(x_n)\}$.

$$\lim_{n\to\infty} f(x_{2n-1}) = \lim_{n\to\infty} (-1)^{2n-1} = \lim_{n\to\infty} (-1) = -1.$$

$$\lim_{n\to\infty} f(x_{2n}) = \lim_{n\to\infty} (-1)^{2n} = \lim_{n\to\infty} 1 = 1.$$

Since $\lim_{n\to\infty} f(x_{2n-1}) \neq \lim_{n\to\infty} f(x_{2n})$, $\lim_{n\to\infty} f(x_n)$ does not exist, by Theorem 4.

So by Theorem 5, $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist.

1. Using the $\in - \delta$ definition prove that

(i) $\lim_{x \to 0} (2x + 3) = 3$

(ii) $\lim_{x \to 1} (2x - 1) = 1$

(iii) $\lim_{x \to -2} (3x + 8) = 2$

(iv) $\lim_{x \to 3} (x^2 + 2x - 8) = 7$

Continuity

Definition

A function f is said to be continuous at a point $a \in D_f$ if

(i) f(x) has definite value f(a) at x = a

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

If one or more of the above conditions fail, the function f is said to be discontinuous at x = a.

The above definition of continuity of a function at a point can also be formulated as follows:

A function f is said to be continuous at x = a if

(i) holds ~~for~~ and for any $\epsilon > 0$, there exists $\delta > 0$ depending on ϵ such that

$|x - a| < \delta \Rightarrow |f(a) - f(x)| < \epsilon$

A function f is continuous on an interval if it is continuous at every point of the interval.

If the interval is a close interval [a,b] the function f is continuous on [a,b] if it is continuous on (a,b), $\lim_{x \to a+} f(x) = f(a)$ and

$\lim_{x \to b-} f(x) = f(b)$.

Example 20 :

Examine the continuity of $|x|$ at $x = 0$.

Solution :

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0. \end{cases}$$

So $\lim_{x \to 0-} |x| = \lim_{x \to 0-} (-x) = 0$

and $\lim_{x \to 0+} |x| = \lim_{x \to 0+} x = 0.$

Since $\lim_{x \to 0-} |x| = \lim_{x \to 0+} |x| = 0$, $\lim_{x \to 0} |x| = 0$.

$|0| = 0$.

Thus $\lim_{x \to 0} |x| = |0|$.

So $|x|$ is continuous at $x = 0$.

Example 21 :

Prove that $[x]$ is discontinuous at $x = n$ for every integer n.

Solution :

In Example 10 of section 11.3 we have seen that $\lim_{x \to n} [x]$ does not exist for any integer n.

So $[x]$ is discontinuous at $x = n$ for every integer n.

The following theorems are consequences of the properties of limit and the definition of continuity of a function at a point.

Theorem 6 :

If the functions f and g are continuous at a point a, then $f + g$, $f - g$, cf (c is a constant), fg are continuous at a. $\frac{f}{g}$ is continuous at a if $g(a) \neq 0$.

Proof :

Since f and g are continuous at a, $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$.

$$\text{So } \lim_{x \to a} (f+g)(x) = \lim_{x \to a} (f(x) + g(x))$$

$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= f(a) + g(a)$$

$$= (f+g)(a).$$

Hence $f + g$ is continuous at a.

We leave the proofs of the other parts as an exercise for the reader.

Theorem 7 :

If a function f is continuous at a point a and a function g is continuous at the point $b = f(a)$, then the composite function gof is continuous at a.

In this case $\lim_{x \to a} (gof)(x) = \lim_{f(x) \to f(a)} g(f(x))$

$$= g(f(a))$$

$$= (gof)(a).$$

The proof of this theorem is the direct application of the definition of limit.

11.8 Monotonic Sequences

Definition :

Let $\{x_n\}$ be a real sequence.

The sequence $\{x_n\}$ is said to be monotonic increasing or non-decreasing if $x_n \leq x_{n+1}$ $\forall\, n \in N$. If $x_n < x_{n+1}$ $\forall\, n \in N$, then the sequence is strictly increasing.

If $x_n \geq x_{n+1}$ $\forall\, n \in N$, then the sequence is monotonic decreasing.

If $x_n > x_{n+1}$ $\forall\, n \in N$, then the sequence is strictly decreasing.

Bounded Sequence

A real sequence $\{x_n\}$ is said to be bounded above if there exists a real number M such that $x_n \leq M$ $\forall\, n \in N$. The number M is called an upper bound of the sequence.

If there exists a real number m such that $x_n \geq m$ $\forall\, n \in N$, then m is called a lower bound of the sequence.

If there exists a real number $K > 0$ such that $|x_n| \leq K$ $\forall\, n \in N$, then the sequence $\{x_n\}$ is said to be bounded.

A sequence $\{x_n\}$ is bounded iff it is bounded below and bounded above,

We need the following theorems for our purpose. The proofs of these theorems appear in higher mathematics.

Theorem 10 :

(a) A monotonic increasing sequence $\{x_n\}$ bounded above is convergent, i.e. x_n tends to a limit as $n \to \infty$.

(b) Monotonic decreasing sequence $\{x_n\}$ bounded below is convergent, i.e. x_n tends to a limit as $n \to \infty$.

Corollary 1 : $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{y \to \infty} \left(1 - \frac{1}{y}\right)^{-y}$

$= \lim_{y \to \infty} \left(\frac{y}{y-1}\right)^y$

$= \lim_{y \to \infty} \left(1 + \frac{1}{y-1}\right)^y$

$= \lim_{y \to \infty} \left\{\left(1 + \frac{1}{y-1}\right)^{y-1} \left(1 + \frac{1}{y-1}\right)\right\}$

$= \lim_{z \to \infty} \left\{\left(1 + \frac{1}{z}\right)^z \left(1 + \frac{1}{z}\right)\right\}$, where $z = y - 1$

$= \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z \times \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)$

$= e \times 1 = e.$

Continuity of an exponential function

A function f defined by $f(x) = a^x \ \forall \ x \in R$ where $a > 0$ and $a \neq 1$ is called an exponential function with base a. We state without proof an important result :

a^x is continuous for every real $x (a > 0, a \neq 1)$.

Corollary : e^x is continuous $\forall \ x \in R$.

Continuity of $\log_a x$, ($a > 0$ and $a \neq 1$) and $x > 0$

For the proof of the continuity of the logarithm function we quote below a theorem whose proof appears in higher mathematics.

Theorem 11 :

If a function defined by $y = f(x)$ is continuous and strictly increasing over an open interval (a, b), then the inverse function defined by $x = f^{-1}(y)$ is strictly increasing and continuous on the range of f.

Remark : The above theorem remains valid if "increasing" is replaced by "decreasing". The theorem also holds if the intervals are infinite intervals.

If $a > 1$, then $y = a^x$ is a continuous and strictly increasing function defined on $R = (-\infty, \infty)$.

The range of this function is $(0, \infty)$. The logarithm function defined by $x = \log_a y$ is the inverse of the function $y = a^x$ with domain $(0, \infty)$ and range $(-\infty, \infty)$. So the logarithm function is strictly increasing and continuous on $(0, \infty)$ if the base a is greater than 1.

If $0 < a < 1$, the argument is quite similar with the word "increasing" replaced by "decreasing".

Thus $\log_a x$ is a strictly increasing and continuous function of x on $(0, \infty)$ by the use of the preceding theorem.

So $\lim\limits_{x \to x_0} \log_a x = \log_a x_0 \ \forall \ x_0 \in (0, \infty)$.

The Continuity of power function x^α

Let us consider the function f defined by $f(x) = x^\alpha$, where α is a real constant. For any given $\alpha \in R$ this function is defined and is positive for $x > 0$.

We have $x^\alpha = e^{\alpha \log_e x}$, where $x > 0$.

Since exponential and logarithm functions are continuous functions it follows that x^α is a continuous function of $x \ \forall \ x > 0$.

If $\alpha > 0$, the function x^α is continuous from the right at $x = 0$.

Since x^α is a continuous function of $x \ \forall \ x > 0$, where α is a real constant

$\lim\limits_{x \to a} x^\alpha = a^\alpha \ \forall \ a > 0$.

Example 25 :

Prove that $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a. \ (a > 0)$

Proof :

Let $a^x - 1 = y$

Then $y \to 0$ as $x \to 0$.

$a^x - 1 = y \Rightarrow a^x = 1 + y$

$\Rightarrow x = \log_a (1 + y)$

$$\text{So } \lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log_a (1 + y)}$$

$$= \lim_{y \to 0} \frac{1}{\frac{\log_a (1 + y)}{y}}$$

$$= \frac{1}{\frac{1}{\ln a}} = \ln a.$$

Cor. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$

Example 33 :

Evaluate $\lim\limits_{x \to 2} \dfrac{\sqrt{x} - \sqrt{2}}{x^4 - 16}$

Solution :

$$\lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x^4 - 16} = \lim_{x \to 2} \frac{\dfrac{\sqrt{x} - \sqrt{2}}{x - 2}}{\dfrac{x^4 - 16}{x - 2}}$$

$$= \lim_{x \to 2} \frac{\dfrac{x^{\frac{1}{2}} - 2^{\frac{1}{2}}}{x - 2}}{\dfrac{x^4 - 2^4}{x - 2}} = \frac{\lim\limits_{x \to 2} \dfrac{x^{\frac{1}{2}} - 2^{\frac{1}{2}}}{x - 2}}{\lim\limits_{x \to 2} \dfrac{x^4 - 2^4}{x - 2}}$$

$$= \frac{\dfrac{1}{2} \times 2^{\frac{1}{2} - 1}}{4 \times 2^{4-1}}$$

$$= \frac{1}{64\sqrt{2}}.$$

Evaluate the following limits :

(i) $\lim_{x \to 0} \frac{x}{\sin 2x}$

(ii) $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x}$

(iii) $\lim_{x \to 0} \frac{\sin mx}{\sin nx}$

(iv) $\lim_{x \to 0} \frac{\tan \alpha x}{x}$

(v) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

(vi) $\lim_{x \to 0} \frac{\sin x^{0}}{x}$